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Variational solution for a cracked mosaic model of woven fabric composites

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Abstract

A variational solution for a cracked mosaic laminate model of woven fabric composites is presented using the principle of minimum complementary energy. The solution is derived for the woven laminate in either the plane strain or the plane stress state, with the warp/fill yarn materials being either orthotropic or transversely isotropic, unlike other existing solutions in the literature of laminate elasticity. The stress components are given in closed-form expressions in terms of a perturbation function, which is governed by two (uncoupled) fourth-order inhomogeneous ordinary differential equations (i.e., Euler–Lagrange equations) when the thermal effects are included. All possible expressions of this perturbation function are obtained in closed forms. Young's modulus of the cracked laminate is calculated using the determined minimum complementary energy. The present closed-form solution can account for different yarn materials, applied loads (crack densities), geometrical dimensions, or their combinations. To demonstrate the solution, a total of 60 sample cases are analyzed using three different composite systems (i.e., glass fiber/epoxy, graphite fiber/epoxy and ceramic fiber/ceramic) and ten different crack densities. The obtained numerical results are also compared to two existing elasticity solutions for cross-ply laminates. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Unidirectional fiber-reinforced composites have been extensively studied and have found widespread structural applications in the last three decades. The knowledge of fabric-reinforced woven (textile) composites, however, is still very limited. Predictions and/or measurements of thermo-elastic properties (including Young's moduli, Poisson's ratios and thermal expansion coefficients) of various woven composites have been the objectives of most studies in this field. A fairly comprehensive review of the relevant works

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published before May 1997 was provided by Tan et al. (1997). More recent advances in predicting the mechanical properties of planar woven composites can be found in Gao and Mall (2000) and references cited therein. On the other hand, very few studies have been reported on modeling of damaged/cracked woven composites, and, as a result, the failure mechanisms of such composites are still not well understood.

Because of their highly complex microstructures, woven composites are very difficult to characterize even experimentally (Roy, 1996, 1998). This necessitates approximations/simplifications of various kinds in their modeling. One host of approximate analyses is based on the recognition that planar woven laminates resemble cross-ply laminates in certain ways. This resemblance enables one to make profitable use of the procedures/results available for cross-ply laminates in studying the behavior of woven laminates. In fact, the mosaic model (Chou and Ishikawa, 1989), one of the earliest analytical models of planar woven composites, regarded the woven composite as an assemblage of asymmetric cross-ply laminate pieces. More recently, special cross-ply laminates were constructed and used as "model laminates" to experimentally study the failure mechanisms of woven composites (Roy, 1996, 1998). Birman and Byrd (1999) also used a cross-ply laminate model to simulate the behavior of a cracked plain-weave ceramic matrix composite. However, there appears to be a lack of analyses based on three-dimensional elasticity theory for cracked woven composites even when using simplified cross-ply laminate models.

The objective of the present study is to provide an analytical solution for estimating stress distributions in and predicting Young's modulus of a cracked mosaic model using the theory of elasticity. The model laminate, which is of cross-ply type, contains four plies (two woven layers) and has the out-of-phase stacking configuration. Both the warp (0° ply) and fill (90° ply) yarns are regarded as orthotropic materials, with transversely isotropic materials being included as a special case. In Section 2, a statically equivalent stress field in terms of an unknown (perturbation) function is constructed, which satisfies all equilibrium equations and traction boundary conditions (including the traction continuity conditions on the interfaces). In this study, the stress components in the thickness direction of the laminate are included, unlike similar studies based on the classical laminate theory. The variational analysis based on the principle of minimum complementary energy is carried out in Section 3 to derive the governing (Euler–Lagrange) equations, which consist of two inhomogeneous fourth-order ordinary differential equations (one for each length segment of the repeating unit). These equations, together with the boundary conditions, define the boundary-value problem to solve for the perturbation function introduced in Section 2. The thermal effects due to the temperature difference are included in this formulation. Section 4 is devoted to the determination of the perturbation function, the stress distributions and Young's modulus. Numerical results of sample cases are given in Section 5. A summary is provided in Section 6 together with discussions on the assumptions/limitations of the new model.

2. Construction of a statically equivalent stress field

Recent experimental studies on planar woven composites under uniaxial tensile loading have shown that the cracking of transverse yarns is the first observable type of damage and other types of damage, including interface debonding and longitudinal yarn cracking, occur only at higher strains in woven composite systems including SiC/SiC (Morvan and Baste, 1998), graphite/epoxy (Roy, 1998) and carbon/polyimide (Gao et al., 1999). Hence, as a first step toward modeling of the damaged woven composites using three-dimensional elasticity theory, a mosaic laminate model with cracked transverse (fill) yarns is adopted in this study. The undamaged mosaic model for predicting mechanical properties of woven composites was first developed by Ishikawa and Chou (Chou and Ishikawa, 1989) using the classical laminate theory.

Consider a four-ply mosaic laminate with the out-of-phase stacking configuration, as shown in Fig. 1.

Based on the experimental observations of Roy (1998) and Gao et al. (1999), it is assumed that the cracks in transverse yarns are of the through-thickness type and are, as a first approximation, uniformly located in

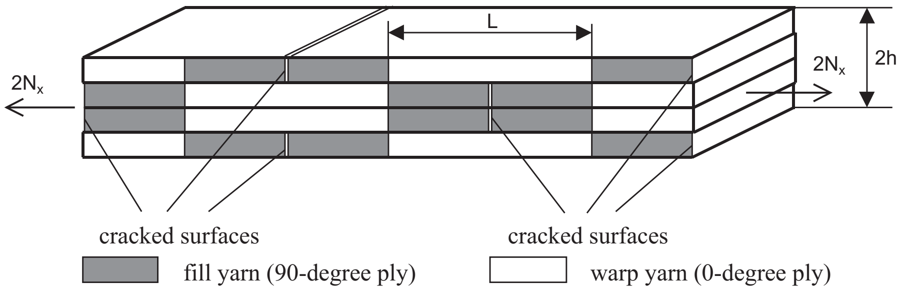

Fig. 1. Mosaic laminate model with cracked transverse (fill) yarns.

the middle of each transverse yarn at the initial stage of damage development under consideration. In addition, we assume that the laminate is in the plane strain or the plane stress state in the z-direction and that the warp and fill yarns are identical, orthotropic (or transversely isotropic) materials. Then, from symmetry and periodicity, only the repeating unit illustrated in Fig. 2 needs to be analyzed. It has the length L, height h and width 1.

Clearly, the boundary conditions for the repeating unit shown in Fig. 2 include the traction-free conditions on the top surface:

$$\sigma_{yy} = \sigma_{xy} = 0 \quad \text{on } y = \frac{h}{2}, \quad -\frac{L}{2} \leqslant x \leqslant \frac{L}{2}, \tag{1a, b}$$

the symmetry condition in the middle plane:

$$\sigma_{xy} = 0 \quad \text{on } y = -\frac{h}{2}, \quad -\frac{L}{2} \leqslant x \leqslant \frac{L}{2}, \tag{2}$$

the traction-free conditions on the crack surfaces:

$$\sigma_{xx} = \sigma_{yx} = 0 \quad \text{on } x = -\frac{L}{2}, \quad 0 \leqslant y \leqslant \frac{h}{2} \text{ and } x = \frac{L}{2}, \quad -\frac{h}{2} \leqslant y \leqslant 0, \tag{3a, b}$$

the traction continuity conditions on the interface $y = 0$:

$$\sigma^{\mathrm{w}}_{yy} = \sigma^{\mathrm{f}}_{yy}, \quad \sigma^{\mathrm{w}}_{xy} = \sigma^{\mathrm{f}}_{xy} \quad \text{on } y = 0, \quad -\frac{L}{2} \leqslant x \leqslant \frac{L}{2}, \tag{4a, b}$$

the traction continuity conditions on the interface $x = 0$:

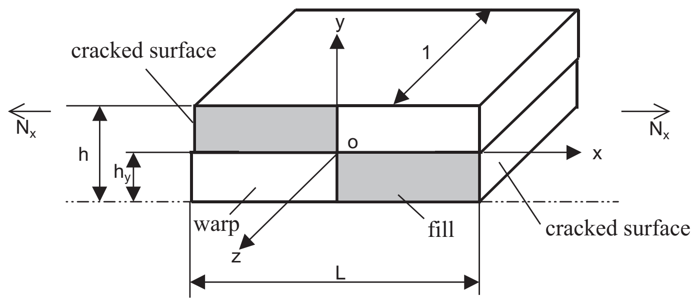

Fig. 2. Repeating unit.

$$\sigma_{xx}^{w} = \sigma_{xx}^{f}, \quad \sigma_{yx}^{w} = \sigma_{yx}^{f} \quad \text{on } x = 0, \ -\frac{h}{2} \leqslant y \leqslant \frac{h}{2}. \tag{5a, b}$$

In addition, the global equilibrium condition is

$$\begin{aligned} N_x &= \int_{-h/2}^{0} \sigma_{xx}^{w} \, dy + \int_{0}^{\frac{h}{2}} \sigma_{xx}^{f} \, dy \quad \left(-\frac{L}{2} \leqslant x \leqslant 0 \right) \\ &= \int_{-h/2}^{0} \sigma_{xx}^{f} \, dy + \int_{0}^{\frac{h}{2}} \sigma_{xx}^{w} \, dy \quad \left(0 \leqslant x \leqslant \frac{L}{2} \right). \end{aligned} \tag{6a, b}$$

In Eqs. (4a,b)–(6a,b) and in the sequel, the superscripts w and f denote warp (0° ply) and fill (90° ply) yarns, respectively. Clearly, the elasticity solution that exactly satisfies the governing field equations and the traction boundary conditions listed above can hardly be obtained analytically. Hence, the principle of minimum complementary energy will be applied to derive an approximate analytical solution here, which allows for certain assumptions on stress distributions. That is, a statically equivalent stress field which satisfies the equilibrium equations and the traction boundary conditions may be constructed to approximate the real stress field that has to satisfy the compatibility equations additionally (Gao and Rowlands, 2000). A general theory for laminate stress analysis was developed by Pagano (1978) using Reissner's variational theorem, which is equivalent to the minimum complementary energy principle when a statically equivalent stress field is invoked.

Along the line of Hashin's (1985) analysis for cracked cross-ply laminates, we assume that for the present mosaic model, the normal stress component in the longitudinal (loading) direction is of the form:

$$\sigma_{xx}^{w} = \sigma_{xx0}^{w} + \varphi(x), \quad \sigma_{xx}^{f} = \sigma_{xx0}^{f} + \psi(x) \qquad \forall x \in \left[-\frac{L}{2}, \frac{L}{2} \right], \tag{7a, b}$$

where $\varphi(x)$, $\psi(x)$ are the perturbations due to the formation of cracks, and

$$\sigma_{xx0}^{w} = \frac{2E_1}{E_1 + E_2} \frac{N_x}{h}, \qquad \sigma_{xx0}^{f} = \frac{2E_2}{E_1 + E_2} \frac{N_x}{h} \tag{8a, b}$$

are the longitudinal normal stress components in the uncracked laminate under tensile load N_x (force/unit length). Here, E_1 and E_2 are Young's moduli of the yarn material along its first (x-axis/z-axis for warp/fill yarns in Fig. 2) and second (z-axis/x-axis for warp/fill yarns in Fig. 2) principal material axes, respectively.

Using Eqs. (7a,b) in Eqs. (6a,b) gives

$$\psi(x) = -\varphi(x) \quad \forall x \in \left[-\frac{L}{2}, \frac{L}{2} \right], \tag{9}$$

where use has been made of the relation

$$\left(\sigma_{xx0}^{w} + \sigma_{xx0}^{f} \right) \frac{h}{2} = N_x. \tag{10}$$

Note that Eqs. (8a,b) and (10) are direct results of the rule of mixtures and the force balance. Eq. (9) shows that there is only one independent perturbation function. Notice that for plane strain or plane stress deformations of orthotropic (or transversely isotropic) materials, the three equilibrium equations reduce to (Gao, 2000)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{11a, b}$$

Using Eq. (7a) in Eqs. (11a,b) yields

$$\begin{cases} \sigma^{w}_{xy} = -\varphi'(x)y + f(x) \\ \sigma^{w}_{yy} = \frac{1}{2}\varphi''(x)y^2 - f'(x)y + g(x) \end{cases} \quad \forall x \in \left[-\frac{L}{2}, 0\right],\ y \in \left[-\frac{h}{2}, 0\right] \text{ and } x \in \left[0, \frac{L}{2}\right],\ y \in \left[0, \frac{h}{2}\right], \tag{12a, b}$$

where $f(x)$ and $g(x)$ are two yet-unknown functions. Similarly, substituting Eq. (7b) into Eqs. (11a,b) gives

$$\begin{cases} \sigma^{f}_{xy} = -\psi'(x)y + F(x) \\ \sigma^{f}_{yy} = \frac{1}{2}\psi''(x)y^2 - F'(x)y + G(x) \end{cases} \quad \forall x \in \left[-\frac{L}{2}, 0\right],\ y \in \left[0, \frac{h}{2}\right] \text{ and } x \in \left[0, \frac{L}{2}\right],\ y \in \left[-\frac{h}{2}, 0\right], \tag{13a, b}$$

where $F(x)$ and $G(x)$ are two additional unknown functions. Next, we enforce the boundary conditions to determine the four unknown functions.

Using Eq. (12a) in Eq. (1b) gives

$$f(x) = \frac{h}{2}\varphi'(x) \quad \forall x \in \left[0, \frac{L}{2}\right], \tag{14}$$

and Eq. (13a) in Eq. (1b) yields

$$F(x) = \frac{h}{2}\psi'(x) \quad \forall x \in \left[-\frac{L}{2}, 0\right]. \tag{15}$$

Inserting Eqs. (12b) and (14) into Eq. (1a) leads to

$$g(x) = \frac{h^2}{8}\varphi''(x) \quad \forall x \in \left[0, \frac{L}{2}\right], \tag{16}$$

and Eqs. (13b) and (15) into Eq. (1a) results in

$$G(x) = \frac{h^2}{8}\psi''(x) \quad \forall x \in \left[-\frac{L}{2}, 0\right]. \tag{17}$$

Similarly, using Eq. (12a) in Eq. (2) gives

$$f(x) = -\frac{h}{2}\varphi'(x) \quad \forall x \in \left[-\frac{L}{2}, 0\right], \tag{18}$$

and Eq. (13a) in Eq. (2) yields

$$F(x) = -\frac{h}{2}\psi'(x) \quad \forall x \in \left[0, \frac{L}{2}\right]. \tag{19}$$

Inserting Eqs. (13b), (15), (17), (18) and (12b) into Eq. (4a) leads to

$$g(x) = \frac{h^2}{8}\psi''(x) \quad \forall x \in \left[-\frac{L}{2}, 0\right], \tag{20}$$

and Eqs. (12b), (14), (16), (19) and (13b) into Eq. (4a) results in

$$G(x) = \frac{h^2}{8}\varphi''(x) \quad \forall x \in \left[0, \frac{L}{2}\right]. \tag{21}$$

Substituting Eqs. (14)–(21) into Eqs. (12a,b) and (13a,b) then gives

$$\begin{cases} \sigma_{xx}^{\mathrm{w}} = \sigma_{xx0}^{\mathrm{w}} + \varphi(x), \\ \sigma_{xy}^{\mathrm{w}} = \varphi'(x)(\frac{h}{2} - y), \\ \sigma_{yy}^{\mathrm{w}} = \frac{1}{2}\varphi''(x)(y - \frac{h}{2})^2, \end{cases} \quad x \in \left[0, \frac{L}{2}\right], \ y \in \left[0, \frac{h}{2}\right], \tag{22a–c}$$

$$\begin{cases} \sigma_{xx}^{\mathrm{w}} = \sigma_{xx0}^{\mathrm{w}} + \varphi(x), \\ \sigma_{xy}^{\mathrm{w}} = -\varphi'(x)(y + \frac{h}{2}), \\ \sigma_{yy}^{\mathrm{w}} = \frac{1}{2}\varphi''(x)[(y + \frac{h}{2})^2 - \frac{h^2}{2}], \end{cases} \quad x \in \left[-\frac{L}{2}, 0\right], \ y \in \left[-\frac{h}{2}, 0\right], \tag{23a–c}$$

$$\begin{cases} \sigma_{xx}^{\mathrm{f}} = \sigma_{xx0}^{\mathrm{f}} - \varphi(x), \\ \sigma_{xy}^{\mathrm{f}} = -\varphi'(x)(\frac{h}{2} - y), \\ \sigma_{yy}^{\mathrm{f}} = -\frac{1}{2}\varphi''(x)(y - \frac{h}{2})^2, \end{cases} \quad x \in \left[-\frac{L}{2}, 0\right], \ y \in \left[0, \frac{h}{2}\right], \tag{24a–c}$$

$$\begin{cases} \sigma_{xx}^{\mathrm{f}} = \sigma_{xx0}^{\mathrm{f}} - \varphi(x), \\ \sigma_{xy}^{\mathrm{f}} = \varphi'(x)(y + \frac{h}{2}), \\ \sigma_{yy}^{\mathrm{f}} = -\frac{1}{2}\varphi''(x)\left[(y + \frac{h}{2})^2 - \frac{h^2}{2}\right], \end{cases} \quad x \in \left[0, \frac{L}{2}\right], \ y \in \left[-\frac{h}{2}, 0\right], \tag{25a–c}$$

where use has also been made of Eqs. (7a,b) and (9). Inserting Eqs. (24a) and (25a) into Eq. (3a) leads to

$$\varphi\left(-\frac{L}{2}\right) = \varphi\left(\frac{L}{2}\right) = \sigma_{xx0}^{\mathrm{f}}, \tag{26a, b}$$

and Eqs. (24b) and (25b) into Eq. (3b) results in

$$\varphi'\left(-\frac{L}{2}\right) = \varphi'\left(\frac{L}{2}\right) = 0. \tag{27a, b}$$

Eq. (4b) is identically satisfied by Eqs. (22b) and (25b) for all $x \in [0, L/2]$, and by Eqs. (23b) and (24b) for all $x \in [-L/2, 0]$. Finally, using Eqs. (22a) and (24a) in Eq. (5a) leads to

$$\varphi(0) = \tfrac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right), \tag{28}$$

and Eqs. (22b) and (24b) in Eq. (5b) results in

$$\varphi'(0) = 0. \tag{29}$$

Eq. (28) is also a consequence of using Eqs. (23a) and (25a) in Eq. (5a), and Eq. (29) a consequence of using Eqs. (23b) and (25b) in Eq. (5b).

Therefore, we have constructed a statically equivalent stress field given by Eqs. (22a–c)–(25a–c) in terms of the unknown perturbation function $\varphi(x)$ that must satisfy Eqs. (26a,b)–(29). Evidently, there exists a large class of such functions which can identically meet the conditions given by Eqs. (26a,b)–(29). Next, we will use the minimum complementary energy principle to determine the right (optimal) one.

3. Variational analysis

In the preceding section, we have constructed a family of statically equivalent stress fields in terms of the unknown function $\varphi(x)$. According to the principle of minimum complementary energy (see, e.g., Washizu, 1982), among all statically equivalent stress fields, the one that makes the total complementary energy Π_c, an absolute (global) minimum, is the actual stress field which also satisfies compatibility. Hence, we will apply this principle to find the right $\varphi(x)$.

Note that for any elastic solid,

$$\Pi_c(\sigma_{ij}) = \int_v W_c(\sigma_{ij})\,\mathrm{d}V - \int_{S_u} T_i u_i^*\,\mathrm{d}S, \tag{30}$$

where W_c is the complementary energy density, T_i is the ith component of the traction vector, u_i^* is the ith component of the prescribed displacement vector on S_u, and V, S_u denote, respectively, the volume and the displacement-prescribed surface part of the elastic solid. For plane deformations with $\varepsilon_{31} = \varepsilon_{32} = \varepsilon_{33} = 0$ (plane strain) or $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ (plane stress),

$$W_c(\sigma_{ij}) = \tfrac{1}{2}[\sigma_{11}\varepsilon_{11}(\sigma_{ij}) + 2\sigma_{12}\varepsilon_{12}(\sigma_{ij}) + \sigma_{22}\varepsilon_{22}(\sigma_{ij})]. \tag{31}$$

Furthermore, if the material is orthotropic (or transversely isotropic), then

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & b_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix} \Delta T, \tag{32}$$

where α_1, α_2 are the thermal expansion coefficients, ΔT is the temperature difference relative to the unloaded (stress-free) state, and b_{ij} are the reduced in-plane compliance constants given by

$$b_{11} = \frac{1 - v_{13}v_{31}}{E_1}, \quad b_{12} = -\frac{v_{12} + v_{13}v_{32}}{E_1}, \quad b_{22} = \frac{1 - v_{23}v_{32}}{E_2}, \quad b_{66} = \frac{1}{G_{12}} \tag{33}$$

for an orthotropic material in the plane strain state, or

$$b_{11} = \frac{1}{E_1}, \quad b_{12} = -\frac{v_{12}}{E_1}, \quad b_{22} = \frac{1}{E_2}, \quad b_{66} = \frac{1}{G_{12}} \tag{34}$$

for an orthotropic material in the plane stress state, or

$$b_{11} = \frac{1}{E_A}\left(1 - \frac{v_A^2 E_T}{E_A}\right), \quad b_{12} = -\frac{v_A(1 + v_T)}{E_A}, \quad b_{22} = \frac{1 - v_T^2}{E_T}, \quad b_{66} = \frac{1}{G_A} \tag{35}$$

for a transversely isotropic material in the plane strain state, or

$$b_{11} = \frac{1}{E_A}, \quad b_{12} = -\frac{v_A}{E_A}, \quad b_{22} = \frac{1}{E_T}, \quad b_{66} = \frac{1}{G_A} \tag{36}$$

for a transversely isotropic material in the plane stress state. In Eqs. (33) and (34), $E_\alpha (\alpha \in \{1, 2\})$, $v_{ij}(i, j \in \{1, 2, 3\})$ and G_{12} are material properties with respect to the three principal material axes, and in Eqs. (35) and (36), the subscripts T and A stand for the transverse plane (yz-plane/xy-plane for warp/fill yarns in this study) and axial direction (x-axis/z-axis for warp/fill yarns in this study), respectively.

Using Eq. (32) in Eq. (31) gives

$$W_c(\sigma_{ij}) = \tfrac{1}{2}\left[b_{xx}\sigma_{xx}^2 + b_{yy}\sigma_{yy}^2 + 2b_{xy}\sigma_{xx}\sigma_{yy} + b_{ss}\sigma_{xy}^2 + (\alpha_x\sigma_{xx} + \alpha_y\sigma_{yy})\Delta T\right], \tag{37}$$

where $\{x, y, z\} \equiv \{1, 2, 3\}$ and $b_{ss} \equiv 1/G_{xy}$. Eq. (37) is applicable to both the orthotropic and transversely isotropic materials in either the plane strain or the plane stress state. In comparison, the corresponding expression given in Hashin (1985) (see Eqs. (2.26a,b) in his paper) is only intended for transversely isotropic materials in the plane stress state and without thermal effects. (McCartney (1992) showed, using a different argument, that Hashin's (1985) solution is for cracked cross-ply laminates with a very small width (and thus in the plane stress state).)

For the present case with no displacement-prescribed boundary part (i.e., $S_u = \text{null}$), Eq. (30) becomes

$$\Pi_c(\sigma_{ij}) = \int_v W_c(\sigma_{ij})\,dV = \int_{-L/2}^{0}\left[\int_{-h/2}^{0} W_c^{w}(\sigma_{ij})\,dy\right]dx + \int_{-L/2}^{0}\left[\int_{0}^{h/2} W_c^{f}(\sigma_{ij})\,dy\right]dx + \int_{0}^{L/2}\left[\int_{-h/2}^{0} W_c^{f}(\sigma_{ij})\,dy\right]dx + \int_{0}^{L/2}\left[\int_{0}^{h/2} W_c^{w}(\sigma_{ij})\,dy\right]dx. \tag{38}$$

Using Eqs. (22a–c)–(25a–c) and (37) in Eq. (38) and carrying out the algebra will result in

$$\Pi_c(\sigma_{ij}) = \int_{-L/2}^{0} J_1(x)\,dx + \int_{0}^{L/2} J_2(x)\,dx, \tag{39}$$

where

$$\begin{aligned} J_1(x) \equiv{}& \frac{h}{4}\left\{b_{xx}^{w}\left[\sigma_{xx0}^{w}+\varphi(x)\right]^2 + b_{xx}^{f}\left[\sigma_{xx0}^{f}-\varphi(x)\right]^2 + \alpha_x^{w}\Delta T\left[\sigma_{xx0}^{w}+\varphi(x)\right] + \alpha_x^{f}\Delta T\left[\sigma_{xx0}^{f}-\varphi(x)\right]\right\} \\ &+ \frac{h^3}{48}\left(b_{ss}^{w}+b_{ss}^{f}\right)\left[\varphi'(x)\right]^2 - \frac{h^3}{48}\left\{5b_{xy}^{w}\left[\sigma_{xx0}^{w}+\varphi(x)\right] + b_{xy}^{f}\left[\sigma_{xx0}^{f}-\varphi(x)\right]\right. \\ &\left.+ \frac{1}{2}\left(5\alpha_y^{w}+\alpha_y^{f}\right)\Delta T\right\}\varphi''(x) + \frac{h^5}{1280}\left(\frac{43}{3}b_{yy}^{w}+b_{yy}^{f}\right)\left[\varphi''(x)\right]^2, \\ J_2(x) \equiv{}& \frac{h}{4}\left\{b_{xx}^{w}\left[\sigma_{xx0}^{w}+\varphi(x)\right]^2 + b_{xx}^{f}\left[\sigma_{xx0}^{f}-\varphi(x)\right]^2 + \alpha_x^{w}\Delta T\left[\sigma_{xx0}^{w}+\varphi(x)\right]\alpha_x^{f}\Delta T\left[\sigma_{xx0}^{f}-\varphi(x)\right]\right\} \\ &+ \frac{h^3}{48}\left(b_{ss}^{w}+b_{ss}^{f}\right)\left[\varphi'(x)\right]^2 + \frac{h^3}{48}\left\{b_{xy}^{w}\left[\sigma_{xx0}^{w}+\varphi(x)\right] + 5b_{xy}^{f}\left[\sigma_{xx0}^{f}-\varphi(x)\right]\right. \\ &\left.+ \frac{1}{2}\left(\alpha_y^{w}+5\alpha_y^{f}\right)\Delta T\right\}\varphi''(x) + \frac{h^5}{1280}\left(b_{yy}^{w}+\frac{43}{3}b_{yy}^{f}\right)\left[\varphi''(x)\right]^2. \end{aligned} \tag{40}$$

Taking the first variation of Eq. (39) and using the essential boundary conditions given in Eqs. (26a,b)–(29) will yield

$$\begin{aligned} \delta\Pi_c ={}& \int_{-L/2}^{0}\left\{\frac{h^5}{640}\left(\frac{43}{3}b_{yy}^{w}+b_{yy}^{f}\right)\varphi^{(4)}(x) - \frac{h^3}{24}\left[\left(b_{ss}^{w}+b_{ss}^{f}\right)+\left(5b_{xy}^{w}-b_{xy}^{f}\right)\right]\varphi''(x)\right. \\ &\left.+ \frac{h}{2}\left(b_{xx}^{w}+b_{xx}^{f}\right)\varphi(x) + \frac{h}{4}\left[2\left(b_{xx}^{w}\sigma_{xx0}^{w}-b_{xx}^{f}\sigma_{xx0}^{f}\right)+\left(\alpha_x^{w}-\alpha_x^{f}\right)\Delta T\right]\right\}\delta\varphi(x)\,dx \\ &+ \int_{0}^{L/2}\left\{\frac{h^5}{640}\left(b_{yy}^{w}+\frac{43}{3}b_{yy}^{f}\right)\varphi^{(4)}(x) - \frac{h^3}{24}\left[\left(b_{ss}^{w}+b_{ss}^{f}\right)+\left(5b_{xy}^{f}-b_{xy}^{w}\right)\right]\varphi''(x) + \frac{h}{2}\left(b_{xx}^{w}+b_{xx}^{f}\right)\varphi(x)\right. \\ &\left.+ \frac{h}{4}\left[2\left(b_{xx}^{w}\sigma_{xx0}^{w}-b_{xx}^{f}\sigma_{xx0}^{f}\right)+\left(\alpha_x^{w}-\alpha_x^{f}\right)\Delta T\right]\right\}\delta\varphi(x)\,dx. \end{aligned} \tag{41}$$

For Π_c to be the global (absolute) minimum, it is required (as a necessary condition) that

$$\delta\Pi_c(\sigma_{ij}) \equiv \delta\Pi_c[\varphi(x)] = 0 \quad \forall\,\varphi(x), x \in \left[-\frac{L}{2},\frac{L}{2}\right]. \tag{42}$$

Using Eq. (41) in Eq. (42) and invoking the fundamental lemma of the calculus of variation will lead to

$$\frac{h^5}{640}\left(\frac{43}{3}b_{yy}^{w}+b_{yy}^{f}\right)\varphi^{(4)}(x) - \frac{h^3}{24}\left[\left(b_{ss}^{w}+b_{ss}^{f}\right)+\left(5b_{xy}^{w}-b_{xy}^{f}\right)\right]\varphi''(x) + \frac{h}{2}\left(b_{xx}^{w}+b_{xx}^{f}\right)\varphi(x) + \frac{h}{4}\left(\alpha_x^{w}-\alpha_x^{f}\right)\Delta T = 0 \qquad \forall x \in \left[-\frac{L}{2},0\right], \tag{43a}$$

$$\frac{h^5}{640}\left(b_{yy}^{\rm w}+\frac{43}{3}b_{yy}^{\rm f}\right)\varphi^{(4)}(x)-\frac{h^3}{24}\left[\left(b_{ss}^{\rm w}+b_{ss}^{\rm f}\right)+\left(5b_{xy}^{\rm f}-b_{xy}^{\rm w}\right)\right]\varphi''(x)$$
$$+\frac{h}{2}\left(b_{xx}^{\rm w}+b_{xx}^{\rm f}\right)\varphi(x)+\frac{h}{4}\left(\alpha_x^{\rm w}-\alpha_x^{\rm f}\right)\Delta T=0 \qquad \forall x\in\left[0,\frac{L}{2}\right], \tag{43b}$$

where use has been made of the fact that $b_{xx}^{\rm w}\sigma_{xx0}^{\rm w}-b_{xx}^{\rm f}\sigma_{xx0}^{\rm f}\equiv 0$ for each of the four sets of constitutive relations listed in Eqs. (33)–(36). These are the governing (Euler–Lagrange) equations for $\varphi(x)$. Note that the two fourth-order ordinary differential equations in Eqs. (43a) and (43b) are different, as in all of the four sets of constitutive relations considered, there are no concurrent relations $b_{yy}^{\rm w}=b_{yy}^{\rm f}$, $b_{xy}^{\rm w}=b_{xy}^{\rm f}$. This is expected, as the two segments (i.e., $x\in[-L/2,0]$ and $x\in[0,L/2]$) of the repeating unit are not symmetric about $x=0$. On the other hand, this differs from the case of a cracked cross-ply laminate (Hashin, 1985), where only one fourth-order ordinary differential equation needs to be solved for the unknown function on the entire interval $x\in[-L/2,L/2]$.

Eqs. (43a) and (43b), together with the (essential) boundary conditions listed in Eqs. (26a,b)–(29), define the boundary value problem (BVP) for determining $\varphi(x)$, $\forall x\in[-L/2,L/2]$. This BVP will be solved in Section 4.

4. Determination of $\varphi(x)$, stress distributions and Young's modulus

The unknown function $\varphi(x)$ will be determined for the two segments separately. To get $\varphi(x)$ defined on $x\in[-L/2,0]$, one needs to solve Eq. (43a) subjected to the following boundary conditions:

$$\varphi\left(-\frac{L}{2}\right)=\sigma_{xx0}^{\rm f},\quad \varphi'\left(-\frac{L}{2}\right)=0,\quad \varphi(0)=\frac{1}{2}\left(\sigma_{xx0}^{\rm f}-\sigma_{xx0}^{\rm w}\right),\quad \varphi'(0)=0, \tag{44}$$

which are initially given in Eqs. (26a), (27a), (28) and (29). Evidently, a particular solution of Eq. (43a) is

$$\varphi_p(x)=-\frac{\left(\alpha_x^{\rm w}-\alpha_x^{\rm f}\right)\Delta T}{2\left(b_{xx}^{\rm w}+b_{xx}^{\rm f}\right)}, \tag{45}$$

which happens to be a constant for given material properties and temperature difference. Eq. (45) shows that the particular solution accounts for the thermal effects. If the thermal term is absent (or neglected), then Eq. (43a) will become homogeneous and $\varphi_p(x)\equiv 0$, as was the case in Hashin (1985).

Note that the homogeneous part of Eq. (43a) can be non-dimensionalized as

$$\varphi^{(4)}(\xi)+p\varphi''(\xi)+q\varphi(\xi)=0 \quad \forall\xi\in\left[-\frac{L}{2h},0\right], \tag{46}$$

where $\xi\equiv x/h$, and

$$\begin{aligned} &p\equiv\frac{a_2}{a_4},\qquad q\equiv\frac{a_0}{a_4},\\ &a_4\equiv\tfrac{1}{320}\left(\tfrac{43}{3}b_{yy}^{\rm w}+b_{yy}^{\rm f}\right),\\ &a_2\equiv-\tfrac{1}{12}\left(b_{ss}^{\rm w}+b_{ss}^{\rm f}+5b_{xy}^{\rm w}-b_{xy}^{\rm f}\right),\\ &a_0\equiv b_{xx}^{\rm w}+b_{xx}^{\rm f}. \end{aligned} \tag{47}$$

Note that $a_0>0$, $a_4>0$ always hold as $b_{xx}^{\rm w}>0$, $b_{yy}^{\rm w}>0$, $b_{xx}^{\rm f}>0$ and $b_{yy}^{\rm f}>0$ (Jones, 1975, p. 43) for all of the four sets of constitutive relations listed in Eqs. (33)–(36). The four roots of the characteristic equation of Eq. (46) are given by

$$r^2 = \sqrt{q}(-m \pm \sqrt{m^2 - 1}), \quad m \equiv \frac{p}{2\sqrt{q}}. \tag{48a, b}$$

Depending on whether $m^2 - 1 > 0$, the four roots have the following different forms:

$$\begin{aligned}
&r_1 = q^{1/4}\left(\sqrt{\frac{1-m}{2}} + \mathrm{i}\sqrt{\frac{1+m}{2}}\right) = \overline{r_3}, \quad r_2 = -q^{1/4}\left(\sqrt{\frac{1-m}{2}} - \mathrm{i}\sqrt{\frac{1+m}{2}}\right) = \overline{r_4}, \quad \text{if } m^2 < 1;\\
&r_1 = \mathrm{i}q^{1/4} = \overline{r_3}, \quad r_2 = -\mathrm{i}q^{1/4} = \overline{r_4}, \quad \text{if } m = 1;\\
&r_1 = q^{1/4} = \overline{r_3}, \quad r_2 = -q^{1/4} = \overline{r_4}, \quad \text{if } m = -1;\\
&r_1 = \mathrm{i}\left(\frac{p}{2}\right)^{1/2}\left(\sqrt{\frac{m-1}{2m}} - \sqrt{\frac{m+1}{2m}}\right) = \overline{r_3},\\
&r_2 = \mathrm{i}\left(\frac{p}{2}\right)^{1/2}\left(\sqrt{\frac{m-1}{2m}} + \sqrt{\frac{m+1}{2m}}\right) = \overline{r_4}, \quad \text{if } m^2 > 1 \text{ and } p > 0;\\
&r_1 = \left(-\frac{p}{2}\right)^{1/2}\left(\sqrt{\frac{m-1}{2m}} - \sqrt{\frac{m+1}{2m}}\right) = -r_3,\\
&r_2 = \left(-\frac{p}{2}\right)^{1/2}\left(\sqrt{\frac{m-1}{2m}} + \sqrt{\frac{m+1}{2m}}\right) = -r_4, \quad \text{if } m^2 > 1 \text{ and } p < 0,
\end{aligned} \tag{49}$$

where the overbar stands for the complex conjugate, and $\mathrm{i} = (-1)^{1/2}$, as usual. These expressions are the same as those presented in Gao (2000), except for the range of m. Then, the homogeneous part of the solution of Eq. (43a), as the general solution of Eq. (46), can be obtained as

$$\begin{aligned}
\varphi_h(x) &= k_1 \cosh\left(\frac{\alpha}{h}x\right)\cos\left(\frac{\beta}{h}x\right) + k_2 \cosh\left(\frac{\alpha}{h}x\right)\sin\left(\frac{\beta}{h}x\right)\\
&\quad + k_3 \sinh\left(\frac{\alpha}{h}x\right)\cos\left(\frac{\beta}{h}x\right) + k_4 \sinh\left(\frac{\alpha}{h}x\right)\sin\left(\frac{\beta}{h}x\right) \quad \text{if } m^2 < 1,\\
\varphi_h(x) &= k_1 \cosh\left(\frac{\lambda}{h}x\right) + k_2 \sinh\left(\frac{\lambda}{h}x\right) + k_3 \frac{x}{h}\cosh\left(\frac{\lambda}{h}x\right) + k_4 \frac{x}{h}\sinh\left(\frac{\lambda}{h}x\right) \quad \text{if } m = -1,\\
\varphi_h(x) &= k_1 \cos\left(\frac{\lambda}{h}x\right) + k_2 \sin\left(\frac{\lambda}{h}x\right) + k_3 \frac{x}{h}\cos\left(\frac{\lambda}{h}x\right) + k_4 \frac{x}{h}\sin\left(\frac{\lambda}{h}x\right) \quad \text{if } m = 1,\\
\varphi_h(x) &= k_1 \cos\left(\frac{\eta-\rho}{h}x\right) + k_2 \sin\left(\frac{\eta-\rho}{h}x\right) + k_3 \cos\left(\frac{\eta+\rho}{h}x\right)\\
&\quad + k_4 \sin\left(\frac{\eta+\rho}{h}x\right) \quad \text{if } m^2 > 1 \text{ and } p > 0,\\
\varphi_h(x) &= k_1 \cosh\left(\frac{\zeta-\gamma}{h}x\right) + k_2 \sinh\left(\frac{\zeta-\gamma}{h}x\right) + k_3 \cosh\left(\frac{\zeta+\gamma}{h}x\right)\\
&\quad + k_4 \sinh\left(\frac{\zeta+\gamma}{h}x\right) \quad \text{if } m^2 > 1 \text{ and } p < 0,
\end{aligned} \tag{50}$$

where

$$\alpha \equiv q^{1/4}\sqrt{\frac{1-m}{2}}, \quad \beta \equiv q^{1/4}\sqrt{\frac{1+m}{2}}, \quad \lambda \equiv q^{1/4}, \quad \eta \equiv \left(\frac{p}{2}\right)^{1/2}\sqrt{\frac{m-1}{2m}},$$
$$\rho \equiv \left(\frac{p}{2}\right)^{1/2}\sqrt{\frac{m+1}{2m}}, \quad \zeta \equiv \left(-\frac{p}{2}\right)^{1/2}\sqrt{\frac{m-1}{2m}}, \quad \gamma \equiv \left(-\frac{p}{2}\right)^{1/2}\sqrt{\frac{m+1}{2m}} \tag{51}$$

are dimensionless parameters, whose values depend solely on material properties, and k_1, k_2, k_3 and k_4 are yet-unknown constants, which are denoted by the same symbols in all five cases for brevity.

The general solution of Eq. (43a) can now be written as

$$\varphi(x) = \varphi_h(x) + \varphi_p(x) \quad \forall\, x \in \left[-\frac{L}{2}, 0\right], \tag{52}$$

with $\varphi_h(x)$ and $\varphi_p(x)$ being given in Eqs. (50) and (45), respectively. The four constants $k_1 - k_4$ involved in the solution can be determined from the four boundary conditions listed in Eq. (44).

For the case with $m^2 < 1$ (i.e., $p^2 < 4q$) the solution gives

$$\varphi(x) = k_1 \cosh\left(\frac{\alpha}{h}x\right)\cos\left(\frac{\beta}{h}x\right) + k_2\left[\cosh\left(\frac{\alpha}{h}x\right)\sin\left(\frac{\beta}{h}x\right) - \frac{\beta}{\alpha}\sinh\left(\frac{\alpha}{h}x\right)\cos\left(\frac{\beta}{h}x\right)\right]$$
$$+ k_4 \sinh\left(\frac{\alpha}{h}x\right)\sin\left(\frac{\beta}{h}x\right) - \frac{(\alpha_x^{\mathrm{w}} - \alpha_x^{\mathrm{f}})\Delta T}{2(b_{xx}^{\mathrm{w}} + b_{xx}^{\mathrm{f}})} \qquad \forall\, x \in \left[-\frac{L}{2}, 0\right], \tag{53}$$

where

$$k_1 = \tfrac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right) - \varphi_p,$$
$$k_4 = \left\langle \left(\alpha^2 + \beta^2\right)\left(\sigma_{xx0}^{\mathrm{f}} - \varphi_p\right)\sinh\left(\frac{\alpha L}{2h}\right)\sin\left(\frac{\beta L}{2h}\right) - \alpha\beta\left[\frac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right) - \varphi_p\right]\right.$$
$$\left. \times \left\{\left[\sinh\left(\frac{\alpha L}{2h}\right)\right]^2 + \left[\sin\left(\frac{\beta L}{2h}\right)\right]^2\right\}\right\rangle \Big/ \left\{\beta^2\left[\sinh\left(\frac{\alpha L}{2h}\right)\right]^2 - \alpha^2\left[\sin\left(\frac{\beta L}{2h}\right)\right]^2\right\}, \tag{54}$$
$$k_2 = \frac{\sigma_{xx0}^{\mathrm{f}} - \varphi_p - \left[\frac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right) - \varphi_p\right]\cosh\left(\frac{\alpha L}{2h}\right)\cos\left(\frac{\beta L}{2h}\right) - k_4\sinh\left(\frac{\alpha L}{2h}\right)\sin\left(\frac{\beta L}{2h}\right)}{\frac{\beta}{\alpha}\sinh\left(\frac{\alpha L}{2h}\right)\cos\left(\frac{\beta L}{2h}\right) - \cosh\left(\frac{\alpha L}{2h}\right)\sin\left(\frac{\beta L}{2h}\right)}.$$

For the case with $m = -1$ (i.e., $p = -2q^{1/2} < 0$), the solution gives

$$\varphi(x) = k_1\cosh\left(\frac{\lambda}{h}x\right) + k_2\left[\sinh\left(\frac{\lambda}{h}x\right) - \lambda\frac{x}{h}\cosh\left(\frac{\lambda}{h}x\right)\right]$$
$$+ k_4\frac{x}{h}\sinh\left(\frac{\lambda}{h}x\right) - \frac{\left(\alpha_x^{\mathrm{w}} - \alpha_x^{\mathrm{f}}\right)\Delta T}{2\left(b_{xx}^{\mathrm{w}} + b_{xx}^{\mathrm{f}}\right)} \qquad \forall x \in \left[-\frac{L}{2}, 0\right], \tag{55}$$

where

$$k_1 = \tfrac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right) - \varphi_p,$$
$$k_4 = \frac{\lambda\left\{\left(\sigma_{xx0}^{\mathrm{f}} - \varphi_p\right)\frac{\lambda L}{2h}\sinh\left(\frac{\lambda L}{2h}\right) - \left[\frac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right) - \varphi_p\right]\left[\sinh\left(\frac{\lambda L}{2h}\right)\right]^2\right\}}{\left[\sinh\left(\frac{\lambda L}{2h}\right)\right]^2 - \left(\frac{\lambda L}{2h}\right)^2}, \tag{56}$$
$$k_2 = -\frac{\lambda\left[\frac{1}{2}\left(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}\right) - \varphi_p\right]\sinh\left(\frac{\lambda L}{2h}\right) + k_4\left[\sinh\left(\frac{\lambda L}{2h}\right) + \frac{\lambda L}{2h}\cosh\left(\frac{\lambda L}{2h}\right)\right]}{\frac{\lambda^2 L}{2h}\sinh\left(\frac{\lambda L}{2h}\right)}.$$

For the case with $m = 1$ (i.e., $p = 2q^{1/2} > 0$), the solution gives

$$\varphi(x) = k_1 \cos\left(\frac{\lambda}{h}x\right) + k_2\left[\sin\left(\frac{\lambda}{h}x\right) - \lambda\frac{x}{h}\cos\left(\frac{\lambda}{h}x\right)\right] + k_4\frac{x}{h}\sin\left(\frac{\lambda}{h}x\right) - \frac{(\alpha_x^{\mathrm{w}} - \alpha_x^{\mathrm{f}})\Delta T}{2(b_{xx}^{\mathrm{w}} + b_{xx}^{\mathrm{f}})} \quad \forall x \in \left[-\frac{L}{2}, 0\right], \tag{57}$$

where

$$k_1 = \tfrac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}) - \varphi_p,$$
$$k_4 = \frac{\lambda\left\{\frac{\lambda L}{2h}(\sigma_{xx0}^{\mathrm{f}} - \varphi_p) - \left[\frac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}) - \varphi_p\right]\sin\left(\frac{\lambda L}{2h}\right)\right\}\sin\left(\frac{\lambda L}{2h}\right)}{\left(\frac{\lambda L}{2h}\right)^2 - \left[\sin\left(\frac{\lambda L}{2h}\right)\right]^2},$$
$$k_2 = -\frac{\lambda\left[\frac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}) - \varphi_p\right]\sin\left(\frac{\lambda L}{2h}\right) - k_4\left[\sin\left(\frac{\lambda L}{2h}\right) + \frac{\lambda L}{2h}\cos\left(\frac{\lambda L}{2h}\right)\right]}{\frac{\lambda^2 L}{2h}\sin\left(\frac{\lambda L}{2h}\right)}. \tag{58}$$

For the case with $m^2 > 1$ and $p > 0$ (i.e., $p > 2q^{1/2} > 0$), the solution gives

$$\varphi(x) = 2k_1 \sin\left(\frac{\eta}{h}x\right)\sin\left(\frac{\rho}{h}x\right) + k_2\left[\sin\left(\frac{\eta-\rho}{h}x\right) - \frac{\eta-\rho}{\eta+\rho}\sin\left(\frac{\eta+\rho}{h}x\right)\right] + \frac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}})\cos\left(\frac{\eta+\rho}{h}x\right) - \frac{(\alpha_x^{\mathrm{w}} - \alpha_x^{\mathrm{f}})\Delta T}{2(b_{xx}^{\mathrm{w}} + b_{xx}^{\mathrm{f}})}\left[1 - \cos\left(\frac{\eta+\rho}{h}x\right)\right] \quad \forall x \in \left[-\frac{L}{2}, 0\right], \tag{59}$$

where

$$k_2 = -\left\langle\left[(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}) - 2\varphi_p\right]\sin\left[\frac{(\eta+\rho)L}{2h}\right]\sin\left(\frac{\eta L}{2h}\right)\sin\left(\frac{\rho L}{2h}\right) + \left\{\sigma_{xx0}^{\mathrm{f}} - \varphi_p - \left[\frac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}) - \varphi_p\right]\cos\left[\frac{(\eta+\rho)L}{2h}\right]\right\}\left\{\frac{\eta-\rho}{\eta+\rho}\sin\left[\frac{(\eta-\rho)L}{2h}\right] - \sin\left[\frac{(\eta+\rho)L}{2h}\right]\right\}\right\rangle \Big/ \left\langle\frac{\eta-\rho}{\eta+\rho}\left\{\sin^2\left[\frac{(\eta-\rho)L}{2h}\right] + \sin^2\left[\frac{(\eta+\rho)L}{2h}\right] + 4\sin^2\left(\frac{\eta L}{2h}\right)\sin^2\left(\frac{\rho L}{2h}\right)\right\} - \left[1 + \left(\frac{\eta-\rho}{\eta+\rho}\right)^2\right]\sin\left[\frac{(\eta-\rho)L}{2h}\right]\sin\left[\frac{(\eta+\rho)L}{2h}\right]\right\rangle,$$
$$k_1 = \left\langle\sigma_{xx0}^{\mathrm{f}} - \varphi_p - \left[\frac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}}) - \varphi_p\right]\cos\left[\frac{(\eta+\rho)L}{2h}\right] + k_2\left\{\sin\left[\frac{(\eta-\rho)L}{2h}\right] - \frac{\eta-\rho}{\eta+\rho}\sin\left[\frac{(\eta+\rho)L}{2h}\right]\right\}\right\rangle \Big/ \left[2\sin\left(\frac{\eta L}{2h}\right)\sin\left(\frac{\rho L}{2h}\right)\right]. \tag{60}$$

For the case with $m^2 > 1$ and $p < 0$ (i.e., $p < -2q^{1/2} < 0$), the solution gives

$$\varphi(x) = k_1\left[\cosh\left(\frac{\zeta-\gamma}{h}x\right) - \cosh\left(\frac{\zeta+\gamma}{h}x\right)\right] + k_2\left[\sinh\left(\frac{\zeta-\gamma}{h}x\right) - \frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left(\frac{\zeta+\gamma}{h}x\right)\right] + \frac{1}{2}(\sigma_{xx0}^{\mathrm{f}} - \sigma_{xx0}^{\mathrm{w}})\cosh\left(\frac{\zeta+\gamma}{h}x\right) - \frac{(\alpha_x^{\mathrm{w}} - \alpha_x^{\mathrm{f}})\Delta T}{2(b_{xx}^{\mathrm{w}} + b_{xx}^{\mathrm{f}})}\left[1 - \cosh\left(\frac{\zeta+\gamma}{h}x\right)\right] \quad \forall x \in \left[-\frac{L}{2}, 0\right], \tag{61}$$

where

$$\begin{aligned} k_2 = &\left\langle \left(\sigma^{\mathrm{f}}_{xx0} - \varphi_p\right)\left\{ \frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\} + \left[\frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p\right] \right. \\ &\left. \times \left\{ \sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\cosh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right]\cosh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\}\right\rangle \\ &\Big/ \left\langle 2\frac{\zeta-\gamma}{\zeta+\gamma}\left\{1 - \cosh\left[\frac{(\zeta-\gamma)L}{2h}\right]\cosh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\} \right. \\ &\left. + \left[1 + \left(\frac{\zeta-\gamma}{\zeta+\gamma}\right)^2\right]\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right]\sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\rangle, \\ k_1 = &\left\langle \sigma^{\mathrm{f}}_{xx0} - \varphi_p - \left[\frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p\right]\cosh\left[\frac{(\zeta+\gamma)L}{2h}\right] \right. \\ &\left. + k_2\left\{\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\}\right\rangle \Big/ \left\{\cosh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \cosh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\}. \end{aligned} \tag{62}$$

This completes the solution of $\varphi(x)$ on $x \in [-L/2, 0]$ for the five different cases.

To get $\varphi(x)$ defined on $x \in [0, L/2]$, one needs to solve Eq. (43b) subjected to the following boundary conditions:

$$\varphi(0) = \frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right), \quad \varphi'(0) = 0, \quad \varphi\left(\frac{L}{2}\right) = \sigma^{\mathrm{f}}_{xx0}, \quad \varphi'\left(\frac{L}{2}\right) = 0, \tag{63}$$

which are initially given in Eqs. (28), (29), (26b) and (27b). By following the same procedures used for determining $\varphi(x)$ on $x \in [-L/2, 0]$, this BVP can be solved.

For the case with $m^2 < 1$ (i.e., $p^2 < 4q$) the solution has the same expression as that given in Eq. (53), but $x \in [0, L/2]$ and the constants k_1, k_2 and k_4 in Eq. (53) need to be replaced by

$$\begin{aligned} k_1 &= \tfrac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p, \\ k_4 &= \left\langle \left(\alpha^2+\beta^2\right)\left(\sigma^{\mathrm{f}}_{xx0} - \varphi_p\right)\sinh\left(\frac{\alpha L}{2h}\right)\sin\left(\frac{\beta L}{2h}\right) - \alpha\beta\left[\frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p\right] \right. \\ &\quad \left. \times \left[\sinh^2\left(\frac{\alpha L}{2h}\right) + \sin^2\left(\frac{\beta L}{2h}\right)\right]\right\rangle \Big/ \left\{\beta^2\left[\sinh\left(\frac{\alpha L}{2h}\right)\right]^2 - \alpha^2\left[\sin\left(\frac{\beta L}{2h}\right)\right]^2\right\}, \\ k_2 &= \frac{\sigma^{\mathrm{f}}_{xx0} - \varphi_p - \left[\frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p\right]\cosh\left(\frac{\alpha L}{2h}\right)\cos\left(\frac{\beta L}{2h}\right) - k_4\sinh\left(\frac{\alpha L}{2h}\right)\sin\left(\frac{\beta L}{2h}\right)}{\cosh\left(\frac{\alpha L}{2h}\right)\sin\left(\frac{\beta L}{2h}\right) - \frac{\beta}{\alpha}\sinh\left(\frac{\alpha L}{2h}\right)\cos\left(\frac{\beta L}{2h}\right)}, \end{aligned} \tag{64}$$

where

$$\alpha \equiv \frac{q^{1/4}}{\sqrt{2}}\sqrt{1 - \frac{p}{2\sqrt{q}}}, \quad \beta \equiv \frac{q^{1/4}}{\sqrt{2}}\sqrt{1 + \frac{p}{2\sqrt{q}}}, \tag{65a, b}$$

$$p \equiv -\frac{80\left(b^{\mathrm{w}}_{ss} + b^{\mathrm{f}}_{ss} - b^{\mathrm{w}}_{xy} + 5b^{\mathrm{f}}_{xy}\right)}{3b^{\mathrm{w}}_{yy} + 43b^{\mathrm{f}}_{yy}}, \quad q \equiv \frac{960\left(b^{\mathrm{w}}_{xx} + b^{\mathrm{f}}_{xx}\right)}{3b^{\mathrm{w}}_{yy} + 43b^{\mathrm{f}}_{yy}}, \tag{65c, d}$$

and φ_p is the same as that given in Eq. (45).

For the case with $m = -1$ (i.e., $p = -2q^{1/2} < 0$), the solution has the same expression as that given in Eq. (55), but $x \in [0, L/2]$ and the constants k_1, k_2 and k_4 in Eq. (55) need to be replaced by

$$k_1 = \tfrac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p,$$
$$k_4 = \frac{\lambda\left\{\left(\sigma^{\rm f}_{xx0} - \varphi_p\right)\frac{\lambda L}{2h}\sinh\left(\frac{\lambda L}{2h}\right) - \left[\frac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p\right]\left[\sinh\left(\frac{\lambda L}{2h}\right)\right]^2\right\}}{\left[\sinh\left(\frac{\lambda L}{2h}\right)\right]^2 - \left(\frac{\lambda L}{2h}\right)^2}, \tag{66}$$
$$k_2 = \frac{\lambda\left[\frac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p\right]\sinh\left(\frac{\lambda L}{2h}\right) + k_4\left[\sinh\left(\frac{\lambda L}{2h}\right) + \frac{\lambda L}{2h}\cosh\left(\frac{\lambda L}{2h}\right)\right]}{\frac{\lambda^2 L}{2h}\sinh\left(\frac{\lambda L}{2h}\right)},$$

where

$$\lambda \equiv \left[\frac{960\left(b^{\rm w}_{xx} + b^{\rm f}_{xx}\right)}{3b^{\rm w}_{yy} + 43b^{\rm f}_{yy}}\right]^{1/4}. \tag{67}$$

For the case with $m = 1$ (i.e., $p = 2q^{1/2} > 0$), the solution has the same expression as that given in Eq. (57), but $x \in [0, L/2]$ and the constants k_1, k_2 and k_4 in Eq. (57) need to be replaced by

$$k_1 = \tfrac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p,$$
$$k_4 = \frac{\lambda\left\{\frac{\lambda L}{2h}\left(\sigma^{\rm f}_{xx0} - \varphi_p\right) - \left[\frac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p\right]\sin\left(\frac{\lambda L}{2h}\right)\right\}\sin\left(\frac{\lambda L}{2h}\right)}{\left(\frac{\lambda L}{2h}\right)^2 - \left[\sin\left(\frac{\lambda L}{2h}\right)\right]^2}, \tag{68}$$
$$k_2 = \frac{\lambda\left[\frac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p\right]\sin\left(\frac{\lambda L}{2h}\right) - k_4\left[\sin\left(\frac{\lambda L}{2h}\right) + \frac{\lambda L}{2h}\cos\left(\frac{\lambda L}{2h}\right)\right]}{\frac{\lambda^2 L}{2h}\sin\left(\frac{\lambda L}{2h}\right)},$$

where λ is given by Eq. (67).

For the case with $m^2 > 1$ and $p > 0$ (i.e., $p > 2q^{1/2} > 0$), the solution has the same expression as that given in Eq. (59), but $x \in [0, L/2]$ and the constants k_1, k_2 in Eq. (59) need to be replaced by

$$\begin{aligned}
k_2 = {} & \Bigg\langle \left[\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - 2\varphi_p\right]\sin\left[\frac{(\eta+\rho)L}{2h}\right]\sin\left(\frac{\eta L}{2h}\right)\sin\left(\frac{\rho L}{2h}\right) \\
& + \left\{\sigma^{\rm f}_{xx0} - \varphi_p - \left[\frac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p\right]\cos\left[\frac{(\eta+\rho)L}{2h}\right]\right\}\left\{\frac{\eta-\rho}{\eta+\rho}\sin\left[\frac{(\eta-\rho)L}{2h}\right] - \sin\left[\frac{(\eta+\rho)L}{2h}\right]\right\}\Bigg\rangle \\
& \Bigg/ \Bigg\langle \frac{\eta-\rho}{\eta+\rho}\left\{\sin^2\left[\frac{(\eta-\rho)L}{2h}\right] + \sin^2\left[\frac{(\eta+\rho)L}{2h}\right] + 4\sin^2\left(\frac{\eta L}{2h}\right)\sin^2\left(\frac{\rho L}{2h}\right)\right\} \\
& - \left[1 + \left(\frac{\eta-\rho}{\eta+\rho}\right)^2\right]\sin\left[\frac{(\eta-\rho)L}{2h}\right]\sin\left[\frac{(\eta+\rho)L}{2h}\right]\Bigg\rangle, \\
k_1 = {} & \Bigg\langle \sigma^{\rm f}_{xx0} - \varphi_p - \left[\frac{1}{2}\left(\sigma^{\rm f}_{xx0} - \sigma^{\rm w}_{xx0}\right) - \varphi_p\right]\cos\left[\frac{(\eta+\rho)L}{2h}\right] \\
& - k_2\left\{\sin\left[\frac{(\eta-\rho)L}{2h}\right] - \frac{\eta-\rho}{\eta+\rho}\sin\left[\frac{(\eta+\rho)L}{2h}\right]\right\}\Bigg\rangle \Bigg/ \left[2\sin\left(\frac{\eta L}{2h}\right)\sin\left(\frac{\rho L}{2h}\right)\right],
\end{aligned} \tag{69}$$

where

$$\eta \equiv \tfrac{1}{2}\sqrt{p - 2\sqrt{q}}, \quad \rho \equiv \tfrac{1}{2}\sqrt{p + 2\sqrt{q}},$$
$$p \equiv -\frac{80\left(b^{\rm w}_{ss} + b^{\rm f}_{ss} - b^{\rm w}_{xy} + 5b^{\rm f}_{xy}\right)}{3b^{\rm w}_{yy} + 43b^{\rm f}_{yy}}, \qquad q \equiv \frac{960\left(b^{\rm w}_{xx} + b^{\rm f}_{xx}\right)}{3b^{\rm w}_{yy} + 43b^{\rm f}_{yy}}, \tag{70}$$

and φ_p is the same as that given in Eq. (45).

Finally, for the case with $m^2 > 1$ and $p < 0$ (i.e., $p < -2q^{1/2} < 0$), the solution has the same expression as that given in Eq. (61), but $x \in [0, L/2]$ and the constants k_1, k_2 in Eq. (61) need to be replaced by

$$\begin{aligned} k_2 = {} & \Bigg\langle \left(\sigma^{\mathrm{f}}_{xx0} - \varphi_p\right)\left\{\frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\} + \left[\frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p\right] \\ & \times \left\{\sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\cosh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right]\cosh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\}\Bigg\rangle \\ & \Bigg/ \Bigg\langle 2\frac{\zeta-\gamma}{\zeta+\gamma}\left\{\cosh\left[\frac{(\zeta-\gamma)L}{2h}\right]\cosh\left[\frac{(\zeta+\gamma)L}{2h}\right] - 1\right\} \\ & - \left[1 + \left(\frac{\zeta-\gamma}{\zeta+\gamma}\right)^2\right]\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right]\sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\Bigg\rangle, \\ k_1 = {} & \Bigg\langle \sigma^{\mathrm{f}}_{xx0} - \varphi_p - \left[\frac{1}{2}\left(\sigma^{\mathrm{f}}_{xx0} - \sigma^{\mathrm{w}}_{xx0}\right) - \varphi_p\right]\cosh\left[\frac{(\zeta+\gamma)L}{2h}\right] - k_2\left\{\sinh\left[\frac{(\zeta-\gamma)L}{2h}\right]\right. \\ & \left.- \frac{\zeta-\gamma}{\zeta+\gamma}\sinh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\}\Bigg\rangle \Bigg/ \left\{\cosh\left[\frac{(\zeta-\gamma)L}{2h}\right] - \cosh\left[\frac{(\zeta+\gamma)L}{2h}\right]\right\}, \end{aligned} \tag{71}$$

where

$$\begin{aligned} & \zeta \equiv \left(-\frac{p}{2}\right)^{1/2}\sqrt{\frac{1}{2} - \frac{\sqrt{q}}{p}}, \quad \gamma \equiv \left(-\frac{p}{2}\right)^{1/2}\sqrt{\frac{1}{2} + \frac{\sqrt{q}}{p}}, \\ & p \equiv -\frac{80\left(b^{\mathrm{w}}_{ss} + b^{\mathrm{f}}_{ss} - b^{\mathrm{w}}_{xy} + 5b^{\mathrm{f}}_{xy}\right)}{3b^{\mathrm{w}}_{yy} + 43b^{\mathrm{f}}_{yy}}, \quad q \equiv \frac{960\left(b^{\mathrm{w}}_{xx} + b^{\mathrm{f}}_{xx}\right)}{3b^{\mathrm{w}}_{yy} + 43b^{\mathrm{f}}_{yy}}, \end{aligned} \tag{72}$$

and φ_p is the same as that given in Eq. (45). This completes the solution of $\varphi(x)$ on $x \in [0, L/2]$ for the five different cases.

Using $\varphi(x)$ determined above in Eqs. (22a–c)–(25a–c) will yield the stress field in the entire laminate (repeating unit). As this stress field is the one that minimizes the total complementary energy and thus is closest to the real stress field (among the family of statically equivalent stress fields constructed in Section 2), it follows from Hashin's (1983) homogenization theorem that the effective Young modulus of the cracked mosaic laminate (as the best lower bound of the real value) is

$$E^{\mathrm{eff}}_x \equiv \frac{\sigma_0^2 L h}{2\Pi^*_{\mathrm{c}}}, \tag{73}$$

where $\sigma_0 \equiv N_x/h$ is the uniform stress applied in the longitudinal direction on the homogenized body, and

$$\Pi^*_{\mathrm{c}} \equiv \min_{\sigma_{ij}} \Pi_{\mathrm{c}}(\sigma_{ij}). \tag{74}$$

Note that rearranging Eqs. (39) and (40) gives, with $b^{\mathrm{w}}_{xx}\sigma^{\mathrm{w}}_{xx0} - b^{\mathrm{f}}_{xx}\sigma^{\mathrm{f}}_{xx0} \equiv 0$,

$$
\begin{aligned}
\Pi_c(\sigma_{ij}) = {} & \frac{hL}{4}\left[b_{xx}^{w}\left(\sigma_{xx0}^{w}\right)^2 + b_{xx}^{f}\left(\sigma_{xx0}^{f}\right)^2 + \left(\alpha_x^{w}\sigma_{xx0}^{w} + \alpha_x^{f}\sigma_{xx0}^{f}\right)\Delta T\right] \\
& + \frac{h}{4}\left(\alpha_x^{w} - \alpha_x^{f}\right)\Delta T \int_{-L/2}^{L/2} \varphi(x)\,\mathrm{d}x + \frac{h}{4}\left(b_{xx}^{w} + b_{xx}^{f}\right)\int_{-L/2}^{L/2}[\varphi(x)]^2\,\mathrm{d}x \\
& + \frac{h^3}{48}\left\{\left(b_{ss}^{w} + b_{ss}^{f} + 5b_{xy}^{w} - b_{xy}^{f}\right)\int_{-L/2}^{0}[\varphi'(x)]^2 dx + \left(b_{ss}^{w} + b_{ss}^{f} + 5b_{xy}^{f} - b_{xy}^{w}\right)\int_{0}^{L/2}[\varphi'(x)]^2\,\mathrm{d}x\right\} \\
& + \frac{h^5}{1280}\left\{\left(\frac{43}{3}b_{yy}^{w} + b_{yy}^{f}\right)\int_{-L/2}^{0}[\varphi''(x)]^2\,\mathrm{d}x + \left(b_{yy}^{w} + \frac{43}{3}b_{yy}^{f}\right)\int_{0}^{L/2}[\varphi''(x)]^2\,\mathrm{d}x\right\}.
\end{aligned}
\tag{75}
$$

Then, using $\varphi(x)$ given by either Eq. (53), Eq. (55), Eq. (57), Eq. (59) or Eq. (61) in Eq. (75) and carrying out the algebra will yield Π_c^* for each of the five cases. With Π_c^* determined, Young's modulus of the cracked mosaic laminate can readily be obtained from Eq. (73). Sample numerical results will be presented in the next section.

5. Numerical results

To illustrate the analytical solution derived in the preceding section, some sample cases are studied in this section, with the relevant numerical results being presented in the table and figure formats.

Consider three different unidirectional composite systems, glass fiber/epoxy, carbon fiber/epoxy and ceramic fiber/ceramic (SiC/1723), as warp/fill yarn materials, whose properties, as reported in Hashin (1985), McCartney (1992) and Ji et al. (1998), are listed in Table 1. The three composites are all treated as transversely isotropic materials.

Based on these material properties, the three fundamental parameters p, q and m can be determined using Eqs. (35), (36), (47), (48b) and (65c,d). The calculated values of p, q and m are tabulated in Table 2.

The value of m together with the sign of p dictates the specific forms of $\varphi(x)$ to be used, as demonstrated in Section 4. From Table 2, it follows that for both the glass/epoxy and graphite/epoxy yarns in either the plane stress or the plane strain state $\varphi(x)$ given in Eqs. (61) and (62) is needed for $x \in [-L/2, 0]$ and $\varphi(x)$ given in Eqs. (53) and (64) for $x \in [0, L/2]$, whereas for the ceramic/ceramic yarns in either the plane stress or the plane strain state $\varphi(x)$ given in Eqs. (53) and (54) should be applied for $x \in [-L/2, 0]$ and $\varphi(x)$ given in Eqs. (53) and (64) for $x \in [0, L/2]$. With the expressions of $\varphi(x)$ identified, the stress components in the cracked laminate can then be easily obtained from Eqs. (22a–c)–(25a–c).

Table 3 lists the values of Young's modulus (E_x^{eff}) of the damaged laminate with different crack densities (h/L). They are calculated using Eqs. (73)–(75), given material properties and the corresponding expressions

Table 1
Material properties

Property	Glass/epoxy (Hashin, 1985)	Graphite/epoxy (Hashin, 1985)	Ceramic/ceramic (Ji et al., 1998)
E_A, GPa	41.7	208.3	140.0
E_T, GPa	13.0	6.5	88.0
ν_A	0.30	0.255	0.20
ν_T	0.42	0.413	0.26
G_A, GPa	3.40	1.65	44.0
G_T, GPa	4.58	2.30	35.0

Table 2
Material parameters

Parameter	Glass/epoxy	Graphite/epoxy	Ceramic/ceramic
Plane stress, $x \in [-L/2, 0]$			
p	−11.5032	−12.4151	−7.2099
q	27.3757	21.5208	33.9876
m	−1.0993	−1.3381	−0.6184
Plane stress, $x \in [0, L/2]$			
p	−8.0965	−8.1886	−5.8087
q	27.3757	21.5208	33.9876
m	−0.7737	−0.8826	−0.4982
Plane strain, $x \in [-L/2, 0]$			
p	−13.4534	−14.7427	−7.4526
q	31.9313	25.5552	35.4300
m	−1.1904	−1.4582	−0.6260
Plane strain, $x \in [0, L/2]$			
p	−8.2314	−8.2845	−5.8091
q	27.6509	21.7605	34.0844
m	−0.7827	−0.8880	−0.4975

of $\varphi(x)$ mentioned above. In these calculations, *Mathematica* program (of Wolfram Research, Inc.) is used to compute the relevant parameters and to numerically evaluate the definite integrals involved in Eq. (75). Also, the numbers involved in the calculations are kept to their ninth decimal place for accuracy.

Note that the values of Young's modulus of the undamaged laminate given in this table (i.e., the column with $L/h = \infty$) is obtained from the rule-of-mixtures formula: $E_{x0}^{\text{eff}} = (E_1 + E_2)/2 \equiv (E_A + E_T)/2$. The data in Table 3 are also illustrated in Figs. 3 and 4, where they are also compared with the known results of Hashin (1985) and McCartney (1992) for cross-ply laminates $[0°/90°]_s$. Note that the Young modulus ratio defined by $r \equiv E_x^{\text{eff}}/E_{x0}^{\text{eff}}$ is used as the ordinate in these two figures.

From Table 3 and Figs. 3 and 4, the following observations can be made:

(1) The degree of reduction in Young's modulus due to the formation of cracks depends on the ratio E_T/E_A of yarn material. For the ceramic/ceramic yarn with the largest value of E_T/E_A, $E_x^{\text{eff}}/E_{x0}^{\text{eff}}$ is the largest (i.e., the reduction degree is the least) among the three, whereas $E_x^{\text{eff}}/E_{x0}^{\text{eff}}$ is the smallest for the graphite/epoxy yarn which has the smallest value of E_T/E_A. This illustrates that ceramic/ceramic woven composite systems are the safest (among the three) to use in damage susceptible environments.

Table 3
Young's modulus E_x^{eff} in GPa

L/h	Glass/epoxy		Graphite/epoxy		Ceramic/ceramic	
	Plane stress	Plane strain	Plane stress	Plane strain	Plane stress	Plane strain
∞	27.35	27.35	107.4	107.4	114.0	114.0
9	24.16	24.88	41.33	41.67	102.61	105.30
8	23.81	24.52	38.36	38.69	101.35	104.01
7	23.36	24.07	35.11	35.42	99.78	102.41
6	22.78	23.47	31.50	31.79	97.76	100.35
5	21.93	22.60	27.42	27.69	95.04	97.57
4	20.54	21.19	22.61	22.87	90.97	93.41
3	17.81	18.44	16.42	16.70	83.33	85.64
2	11.61	12.18	8.14	8.41	62.86	64.83
1	2.07	2.26	1.08	1.16	14.02	14.62

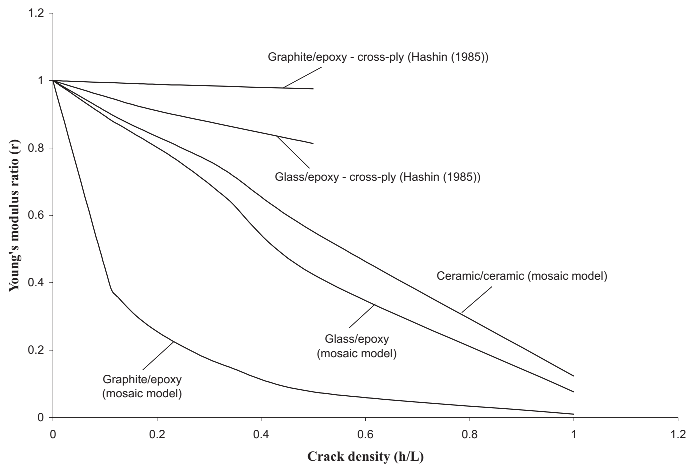

Fig. 3. Plane stress laminate.

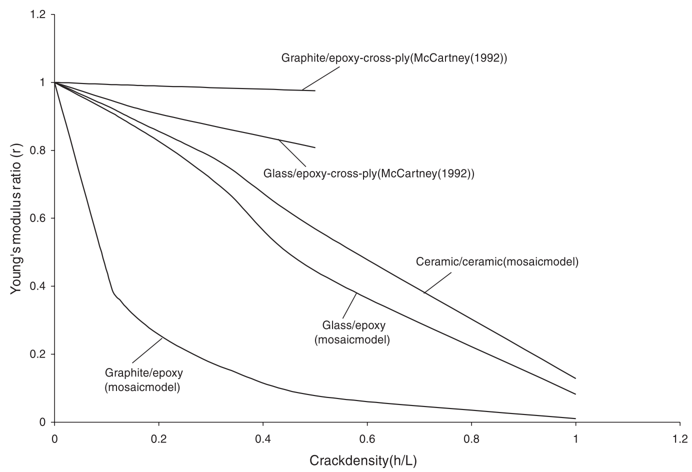

Fig. 4. Plane strain laminate.

(2) A larger reduction in Young's modulus occurs for the mosaic laminate than for the cross-ply laminates with the same crack density. This agrees with the fact that in-plane properties of planar woven composites are weaker than those of laminated composites.

(3) The values of E_x^{eff} in the plane strain case are always larger than those in the plane stress case. That is, more damages occur in the plane stress mosaic laminate than in the plane strain one under the same applied stress. This is consistent with other damage analyses based on elasticity theory. For example, it is known in linear elastic fracture mechanics that the plastic (damage) zone near a Mode I crack tip in the plane strain state is smaller than that in the plane stress state, and the same is true for the plastic (damage) zone under a concentrated normal/shear force acting on a half plane (Gao, 1999) in contact mechanics. However, the differences between the two sets of values are consistently small. This implies that the conservative results from the plane stress analyses can be adopted to represent typical problems with finite width. In other words, the use of a plane stress strip model to characterize the behavior of cracked planar woven composites is justified.

6. Summary

A variational solution based on the principle of minimum complementary energy is presented for a cracked mosaic laminate model of woven fabric composites. The model laminate consists of two woven layers (four plies) in an out-of-phase stacking configuration. The solution is quite general and can accommodate the laminate in either the plane strain or the plane stress state, with the warp/fill yarn materials being either orthotropic or transversely isotropic. This differs from other existing solutions in the literature of laminate elasticity. The only assumption used in constructing the statically equivalent stress field is that the stress component in the loading direction is independent of the thickness coordinate, as was done in Hashin (1985) and McCartney (1992) for cross-ply laminates. The stress components are derived explicitly in terms of a perturbation function, which is governed by a fourth-order ordinary differential equation in each of the two segments of the repeating unit. The two ordinary differential equations are homogeneous only when the thermal effects are absent (neglected). All possible expressions of this perturbation function are obtained in closed forms, which one to be used depends on three material parameters. The total minimum complementary energy and thus Young's modulus of the cracked laminate are determined using the identified expression(s) of the perturbation function directly. *Mathematica* program of the Wolfram Research, Inc. is used to compute various parameters and to numerically evaluate the definite integrals involved in the complementary energy expression.

Being derived in a closed form, the present solution can naturally account for different yarn materials, applied loads (crack densities), geometrical dimensions, or their combinations. To demonstrate the solution, a total of 60 sample cases are analyzed using three different composite systems (i.e., glass fiber/epoxy, graphite fiber/epoxy and ceramic fiber/ceramic) and ten different crack densities. The calculations are carried out using non-dimensional quantities, with L/h being the only geometrical parameter. The obtained numerical results are also compared to Hashin's (1985) plane stress and McCartney's (1992) plane strain solutions for cross-ply laminates, which shows consistency among the three different theoretical models. A comparison with suitable experimental data would definitely enhance the present analysis. Unfortunately, the inherent difficulty in experimentally modeling woven composites (Roy, 1996, 1998; Tan et al., 1997) has made experimental data on mechanical properties of damaged woven composites extremely scarce. This prevented us from finding comparable experimental data and including the desired comparison.

As always, the present analysis has its own limitations, which arise from the assumptions used. First of all, the mosaic model neglects the undulations of yarns. Consequently, the newly developed model may only be good for analyzing cracked woven composites with very small (negligible) undulation lengths. Of course, the closed-form solutions derived here for the mosaic model, as idealized as the model is, provide benchmarks for the validation of various numerical models/computer codes that are usually applied to numerically solve problems of woven composites involving more complicated geometries and/or damage patterns. In addition, the present mosaic model, as defined in Fig. 1, is a series model (Chou and Ishikawa, 1989) with cracked transverse yarns, and, as a result, the predicted values of Young's modulus of the model laminate are expected to be larger than those of the series model, but smaller than those of the parallel model, of the corresponding woven laminate with undulations (Naik, 1994). Finally, the assumption that there is only one crack in the middle of each transverse yarn is another idealization. Some woven composite systems may not exhibit the assumed damage pattern, although this assumption is based on the experimental observations reported in Morvan and Baste (1998) and Gao et al. (1999) for the early stage of damage development in the tested woven composite materials. Similar situations exist in the damage modeling of cross-ply laminates, which are the prototypes of the mosaic model used here. It is therefore suggested that extra caution should be exercised on damage modeling of woven composites, including the use of the newly proposed model, which is perhaps the simplest analytical model for damaged woven composites based on elasticity theory. It is hoped that the present idealized model will pave the way for the development of more sophisticated models that can account for the actual geometry and/or the real damage pattern of a woven composite. In fact, another model including yarn undulations has been under development as the completion of this work and will be reported elsewhere.

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